The Pigeonhole Principle

1. State the pigeonhole principle; then state the contrapositive of the pigeonhole principle. Which seems more natural?

2. Show that at least two students in Math 184 were born on the same day of the month, and that at least five were born on the same day of the week.

3. A *blue moon* is the rare occurrence when there are two new moons in the same month. Show that there is at least one blue moon every year. (You can assume the lunar cycle is exactly 28 days).

4. Consider a 4-by-82 square grid of points, each of which is colored red, green, or blue. Prove that there exists some rectangle with sides parallel to the gridlines whose corners are all the same color (How many different ways are there to color a row of four points?). How large a grid would you need to make the same argument work if there are n colors?

5. Let E be a nonempty finite subset of the plane. Then we can define the *distance set* $\Delta(E)$ to be

$$\Delta(E) = \{||x - y|| : x, y \in E, x \neq y\}$$

the set of distances between points of E. Clearly $|\Delta(E)| \leq |E|^2$; the famous combinatorialist Paul Erdős proved in 1946 that $|\Delta(E)| \geq C\sqrt{|E| - 1}$ for some $C > 0$. Reconstruct his argument using the following outline:

   - Pick any point $x \in E$, and draw all possible circles centered at x passing through other points of E. Then either there are at least $\sqrt{|E| - 1}$ circles, or at least one circle passes through at least $\sqrt{|E| - 1}$ points. In either case there are at least $\frac{1}{2}\sqrt{|E| - 1}$ distances.

   Erdős then conjectured that $|\Delta(E)|$ grows faster than $|E|^\alpha$ for any $\alpha < 1$; this was proven by Guth and Katz in 2015.

Abstract algebra

6. Let K be a finite field. Show that every element of K is the sum of two squares. (You don't need any special knowledge about fields besides the fundamental theorem of algebra. *Hint:* How many elements of K are squares?)