- The famous combinatorialist Gian-Carlo Rota proposed the following series of twelve basic counting problems in combinatorics, which are known as the *twelvefold way*. How many ways are there to place n balls in k boxes if

  A. we treat all the boxes and balls as being distinct

  B. the balls are distinct, but the boxes are not

  C. the boxes are distinct, but the balls are not

  D. neither the balls nor the boxes are distinct

  and we impose the condition that

  1. (no additional condition)?
  2. every box gets at least one ball?
  3. every box gets at most one ball?

  Combining each of the four settings above with one of the three conditions below gives twelve problems. Note that the solutions for D1 and D2 rely on the *partition function*, which we have not talked about; so you don't have to worry about doing those.

  One main takeaway from this is that counting labeled objects is usually easier than counting unlabeled ones.

- Here is a classic use of the inclusion/exclusion principle. For positive integers n, Euler's totient function $\phi(n)$ is defined as the number of integers from 1 to n which are relatively prime to n. For instance, $\varphi(12) = 4$ because the only integers in [12] which are relatively prime to 12 are $1, 5, 7$, and 11. This function is very important in number theory.

  1. Find $\varphi(77)$. (Hint: Which integers are not relatively prime to 77?)
  2. Find $\varphi(100)$.
  3. Find $\varphi(2 \cdot 3 \cdot 5 \cdot 7)$.
  4. Show that $\varphi(666) = 6 \cdot 6 \cdot 6$.